

AMENABILITY AND REPRESENTATION THEORY OF PRO-LIE GROUPS

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ABSTRACT. We develop a semigroup approach to representation theory for pro-Lie groups satisfying suitable amenability conditions. As an application of our approach, we establish a one-to-one correspondence between equivalence classes of unitary irreducible representations and coadjoint orbits for a class of pro-Lie groups including all connected locally compact nilpotent groups and arbitrary infinite direct products of nilpotent Lie groups. The usual C^* -algebraic approach to group representation theory positively breaks down for infinite direct products of non-compact locally compact groups, hence the description of their unitary duals in terms of coadjoint orbits is particularly important whenever it is available, being the only description known so far.

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1. INTRODUCTION

Pro-Lie groups are topological groups that are isomorphic to projective limits of finite-dimensional Lie groups, and their structure theory and Lie theory were developed to an impressive extent in the monograph [HM07]. Every connected locally compact group but also every infinite direct product of finite-dimensional Lie groups belong to that remarkable class of topological groups. The precise relation between pro-Lie groups and infinite-dimensional Lie groups was also investigated in [HN09]. Irreducible representation theory of pro-Lie groups was perhaps less developed so far, so in the present paper we try to fill that gap, following the pattern of the method of coadjoint orbits from representation theory of nilpotent Lie groups (see Theorem 4.6 below). It is maybe unexpected that a substantial part of our investigation can be developed on the level of semitopological semigroups, and their amenability properties actually play a key role in our approach.

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To describe our results in more detail, let G be a topological group with Lie algebra $\mathbf{L}(G)$ as in [HM07]. One has the coadjoint action $\text{Ad}_G^*: G \times \mathbf{L}(G)^* \rightarrow \mathbf{L}(G)^*$, with the corresponding set of coadjoint orbits denoted by $\mathbf{L}(G)^*/G$. Also denote by \widehat{G} the set of equivalence classes of unitary irreducible representations of G . In the spirit of the orbit method, there should be a (at least partially defined) correspondence $\Psi_G: \widehat{G} \rightarrow \mathbf{L}(G)^*/G$. Existence of Ψ_G for pretty large classes of pro-Lie groups is established in Theorem 4.6 below, and the coadjoint orbits in the image of Ψ_G will be said to satisfy the *integrality condition*. The study of these orbits beyond the usual setting of Lie groups is a major motivation of the present paper. And it turns out that the key to reducing the study of Ψ_G from topological groups to Lie groups is the amenability. Additional motivation for us is the study of separation properties of unitary representations in terms of suitable moment sets, which we plan to extend from Lie groups (see [ASZ11] and [Ze11]) to more general topological groups, using the methods developed here and the moment sets introduced in [BNi15].

The above integrality terminology claims its origin in algebraic topology (cohomology with integer coefficients), but for our purposes here it is more relevant to recall that if $G = \mathbb{T}$ is the unit circle regarded as a compact Lie group in the usual way, then its coadjoint orbits are the singleton sets $\{t\}$ with $t \in \mathbb{R}$, and the integrality condition on $\{t\}$ essentially means $t \in \mathbb{Z}$. This is a special instance of the duality theory for locally compact abelian groups. In fact, every topological abelian group has a Lie algebra, and its coadjoint orbits are the singleton sets $\{\xi\}$ with $\xi \in \mathbf{L}(G)^*$. Some non-locally-compact abelian groups have only one unitary representation, namely the trivial representation (see for instance [Ba91]), hence only one orbit satisfying the integrality condition, namely $\{0\}$. However, in the locally compact case, the Pontryagin duality clarifies that there exist many coadjoint orbits that satisfy the integrality condition, and they are parameterized by the dual group. It is very interesting that although coadjoint orbits were not explicitly mentioned in connection with the duality theory of locally compact abelian groups, one of the most natural approaches to that theory is based on approximation with abelian Lie groups via projective limits (see for instance [Mr77]), and the dual object of an abelian Lie group is directly related to coadjoint orbits.

We investigate the above ideas beyond the case of abelian groups, in the spirit of the Kirillov theory on nilpotent Lie groups (see [Ki62]). We wish to study classes \mathcal{C} of topological groups G for which there exists a map $\Psi_G: \widehat{G} \rightarrow \mathbf{L}(G)^*/G$ that is equal to the Kirillov correspondence if G is a connected nilpotent Lie group and is functorial: If H is another topological group in the class \mathcal{C} and $p: G \rightarrow H$ is any continuous surjective homomorphism, then one has the commutative diagram

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\widehat{p}} & \widehat{G} \\ \Psi_H \downarrow & & \downarrow \Psi_G \\ \mathbf{L}(H)^*/H & \xrightarrow{\check{p}} & \mathbf{L}(G)^*/G \end{array}$$

where \widehat{p} and \check{p} are the maps that are canonically associated to p . This abstract approach is applied here to representation theory of pro-Lie groups, thus investigating to what extent the method of coadjoint orbits carries over via projective limits beyond the framework of Lie groups. This study is strongly motivated by [HM07, Ch. 9, Postscript].

The structure of this paper is as follows. In Section 2 we collect some terminology, notation, and a couple of basic examples of pro-Lie groups to which our methods are particularly well suited:

- Infinite direct products of connected nilpotent Lie groups.
- Connected nilpotent locally compact groups.

In Section 3 we develop our semigroup amenability approach to factor representations, the main technical here being Theorem 3.4. We also establish here in Proposition 3.8, for later use, amenability of any infinite direct product of amenable topological groups, a result that we were not able to locate in the existing literature, although it has been long known that it fails to be true for discrete groups [Da57, page 517]. In Section 4 we establish our main result on the bijective correspondence between the unitary dual and the integral coadjoint orbits of pro-Lie groups that satisfy suitable amenability conditions (Theorem 4.6). We then specialize it for the aforementioned two classes of groups (Corollaries 4.7 and 4.9) and we also discuss some specific examples. As the connected nilpotent locally compact groups are projective limits of nilpotent Lie groups that may not be simply connected, (see Examples 2.4 and 4.8 below), we also need a few results from representation theory of these Lie groups and we collected these with full proofs in Appendix A, because again they do not seem to be easily accessible in the earlier literature.

2. PRELIMINARIES

In this section we record some terminology and notation to be used throughout this paper. We recall that a monoid is a semigroup with unit element, and a submonoid is any subsemigroup that contains the unit element. We denote by $\mathbf{1}$ the unit element of any monoid whose composition law is denoted multiplicatively, as well as the identity map of any vector space. If S is any semigroup, then a *normal subsemigroup* of S is any subset $N \subseteq S$ such that $NN \subseteq N$ and $sN = Ns$ for every $s \in S$. In this case we can define an equivalence relation on S by

$$s_1 \sim s_2 \iff s_1 N = s_2 N$$

and the corresponding quotient set $S/N := \{sN \mid s \in S\}$ has the natural structure of a semigroup with the operation $sN \cdot tN := stN$. It is straightforward to check that this operation on S/N is well defined and the quotient map

$$p_N: S \rightarrow S/N, \quad s \mapsto sN$$

is a homomorphism of semigroups. If S is a monoid or a group, then so is S/N .

A homomorphism of semigroups $S_1 \rightarrow S_2$ is said to be normal if its image is a normal subsemigroup of S_2 .

Now assume that the semigroup S is equipped with a topology. We say that S is a *right* (respectively, *left*) *topological semigroup* if for each $s \in S$ the mapping $S \rightarrow S, t \mapsto ts$ (respectively, $t \mapsto st$) is continuous. Moreover S is a *semitopological semigroup* if it is both left and right topological.

Notation 2.1. For every complex Hilbert space \mathcal{H} we denote by $\mathcal{B}(\mathcal{H})$ its set of all continuous linear operators. Besides the operator norm topology, we will use the strong operator topology and the strong* operator topology on $\mathcal{B}(\mathcal{H})$, defined as follows. If $\{a_j\}_{j \in J}$ is a set in $\mathcal{B}(\mathcal{H})$ and $a \in \mathcal{B}(\mathcal{H})$, then $\lim_{j \in J} a_j = a$ in the strong operator topology if $\lim_{j \in J} \|a_j \xi - a \xi\| = 0$ for every $\xi \in \mathcal{H}$, while $\lim_{j \in J} a_j = a$ in the

strong* operator topology if we have both $\lim_{j \in J} a_j = a$ and $\lim_{j \in J} a_j^* = a^*$ in the strong operator topology. The strong operator topology and the strong* operator topology coincide on the unitary group

$$U(\mathcal{H}) := \{u \in \mathcal{B}(\mathcal{H}) \mid u^*u = uu^* = \mathbf{1}\}$$

and turn $U(\mathcal{H})$ into a topological group.

The contraction monoid of \mathcal{H} is defined as $C(\mathcal{H}) := \{a \in \mathcal{B}(\mathcal{H}) \mid \|a\| \leq 1\}$. It is well known that the operator multiplication map $C(\mathcal{H}) \times C(\mathcal{H}) \rightarrow C(\mathcal{H})$, $(a, b) \mapsto ab$, is separately continuous with respect to any of the strong operator topology and the strong* operator topology, hence any of these two topologies turn $C(\mathcal{H})$ into a semitopological monoid.

For any monoid S , a *filter basis of submonoids* is a set $\mathcal{N} \neq \emptyset$ of submonoids with the property that for every $N_1, N_2 \in \mathcal{N}$ there exists $N_3 \in \mathcal{N}$ with $N_3 \subseteq N_1 \cap N_2$. If moreover S is equipped with a topology, then we say that \mathcal{N} is *converging to the identity* if for every neighborhood V of $\mathbf{1} \in S$ there exists $N \in \mathcal{N}$ with $N \subseteq V$.

Remark 2.2. We say that the topological group G is *complete* if every Cauchy net in G is convergent, where a Cauchy net is any family $\{g_j\}_{j \in J}$ of elements of G , where J is a directed set, such that for every neighborhood V of $\mathbf{1} \in G$ there exists $i \in J$ such that $g_j g_k^{-1} \in V$ for all $j, k \in J$ with $j, k \geq i$. Every locally compact group is complete by [HM07, Rem. 1.31] and then so is every projective limit of Lie groups by [HM07, Lemma 1.32(iii)]. We also recall from [HM07, Th. 1.33] that if G is a complete topological group with a filter basis \mathcal{N} of closed normal subgroups converging to the identity, then the map

$$G \rightarrow \prod_{N \in \mathcal{N}} G/N, \quad g \mapsto (p_N(g))_{N \in \mathcal{N}}$$

gives an isomorphism of topological groups $\gamma: G \rightarrow \varprojlim_{N \in \mathcal{N}} G/N$.

In this setting, let us establish some further notation to be used throughout this paper, unless otherwise mentioned. Assume that for every $N \in \mathcal{N}$ the topological group G/N is a Lie group. If $N_1, N_2 \in \mathcal{N}$ with $N_1 \subseteq N_2$, then the map

$$p_{N_1, N_2}: G/N_1 \rightarrow G/N_2, \quad gN_1 \mapsto gN_2$$

is a well-defined continuous homomorphism of Lie groups, hence p_{N_1, N_2} is automatically smooth. Thus the family $\{G/N\}_{N \in \mathcal{N}}$ is organized as a projective system of Lie groups, whose projective limit is isomorphic to the locally compact group G (see [HM07, Th. 1.30, 1.33]). We write $G = \varprojlim_{N \in \mathcal{N}} G/N$.

In this way the topological group G is *approximated* by the Lie groups G/N with $N \in \mathcal{N}$.

We now briefly indicate two examples of groups whose representation theory will be studied in Examples 4.7 and 4.9, respectively.

Example 2.3. Let $\{G_j\}_{j \geq 1}$ be any sequence of Lie groups. Their Cartesian product $G := \prod_{j \geq 1} G_j$ is a topological group, which in general is not a Lie group. For every $k \geq 0$ define

$$N_k := \prod_{j \geq k+1} G_j \simeq \underbrace{\{1\} \times \cdots \times \{1\}}_{k \text{ times}} \times G_{k+1} \times G_{k+2} \times \cdots \subseteq G.$$

Thus every N_k is a closed normal subgroup of G and $G/N_k \simeq G_1 \times \cdots \times G_k$ is a Lie group. Then it is easily checked that the family $\mathcal{N} := \{N_k \mid k \geq 0\}$ satisfies the above conditions in Remark 2.2, hence the topological group G is the projective limit of the sequence of Lie groups $\{G_1 \times \cdots \times G_k\}_{k \geq 1}$.

Example 2.4. Let G be any connected Lie group with a sequence of discrete central subgroups

$$G \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots$$

with $\bigcap_{k \geq 1} \Gamma_k = \{1\}$. Then we have the projective system of Lie groups

$$G/\Gamma_1 \rightarrow G/\Gamma_2 \rightarrow \cdots$$

which generalizes the solenoids discussed in [Cz74].

Remark 2.5. Every *connected* locally compact group G admits a family \mathcal{N} as in Remark 2.2, for instance the family of *all* its compact normal subgroups $N \subseteq G$ for which the locally compact group G/N is a Lie group. This fact is known as Yamabe's theorem (see for instance [BNi15, Lemma 2.3]).

3. FACTOR REPRESENTATIONS OF SEMIGROUPS AND PROJECTIVE LIMITS OF GROUPS

In this section we establish some general results in representation theory of semitopological monoids, with applications to representations of projective limits, motivated by the following facts. Let G be any almost connected locally compact group and denote by $\mathcal{N}_0(G)$ its family of compact normal subgroups $N \subseteq G$ for which G/N is a Lie group. By [Lip72, Th. 2.1], $G = \varprojlim_{N \in \mathcal{N}_0(G)} G/N$ implies $\widehat{G} = \varinjlim_{N \in \mathcal{N}_0(G)} \widehat{G/N}$,

and in particular

$$\widehat{G} = \bigcup_{N \in \mathcal{N}_0(G)} \widehat{G/N}$$

where $\widehat{G/N}$ is the image of $\widehat{G/N}$ in \widehat{G} under the injective map $\widehat{p}_N : \widehat{G/N} \rightarrow \widehat{G}$. The dual space \widehat{G} of equivalence classes of irreducible unitary representations, is thus realized as an inductive limit of the dual spaces of Lie groups. Here we generalize the above result in several directions, inasmuch as we study factor representations of not necessarily compact semitopological monoids, rather than irreducible representations of locally compact groups.

For any semigroup S we denote by $\ell^\infty(S)$ the commutative unital C^* -algebra of all complex bounded functions on S with the sup norm $\|\cdot\|_\infty$. For each $t \in S$ we define

$$L_t : \ell^\infty(S) \rightarrow \ell^\infty(S) \quad \text{and} \quad R_t : \ell^\infty(S) \rightarrow \ell^\infty(S)$$

by $(L_t f)(s) = f(ts)$ and $(R_t f)(s) = f(st)$ whenever $s \in S$ and $f \in \ell^\infty(S)$.

If the semigroup S is equipped with a topology then we denote by $\mathcal{C}_b(S)$ the set of all continuous functions in $\ell^\infty(S)$. When S is a semitopological semigroup we denote $\mathcal{LUC}_b(S)$ the set of all *left uniformly continuous* bounded complex functions on S . That is, $f \in \mathcal{LUC}_b(S)$ if and only if $f \in \mathcal{C}_b(S)$ and the mapping $S \rightarrow \mathcal{C}_b(S)$, $s \mapsto L_s f$, is continuous. Similarly, we define the set $\mathcal{RUC}_b(S)$ of all *right uniformly continuous* bounded complex functions on S by the above condition with L_s replaced by R_s . Moreover, we introduce the set $\mathcal{UC}_b(S) := \mathcal{LUC}_b(S) \cap \mathcal{RUC}_b(S)$ consisting of all

uniformly continuous bounded complex functions on S . It is clear that all of the sets $\mathcal{LUC}_b(S)$, $\mathcal{RUC}_b(S)$ and $\mathcal{UC}_b(S)$ are unital C^* -subalgebras of $\mathcal{C}_b(S)$.

Next denote $\mathcal{T} := \mathcal{LUC}_b(S)$. A *mean* on S is a linear functional $\mu: \mathcal{T} \rightarrow \mathbb{C}$ satisfying $0 \leq \mu(\varphi)$ if $0 \leq \varphi \in \mathcal{T}$, and $\|\mu\| = \mu(\mathbf{1}) = 1$, where $\mathbf{1}$ is the constant function equal to 1 on G . It is well known that this implies $\operatorname{Re} \mu(\varphi) = \mu(\operatorname{Re} \varphi)$ for every $\varphi \in \mathcal{T}$. Now note that \mathcal{T} is invariant under the operators L_t for each $t \in S$. We say that a mean $\mu: \mathcal{T} \rightarrow \mathbb{C}$ is *left invariant* if $\mu \circ L_t = \mu$ for all $t \in S$. If this is the case, then S is said to be *amenable*. For instance, every compact group is amenable, and a left invariant mean is given by its probability Haar measure, suitably normalized. Also, every solvable topological group is amenable (see [Da57]).

Lemma 3.1. *Let \mathcal{H} be a complex Hilbert space with its contraction semigroup $C(\mathcal{H})$ endowed with the strong operator topology. Let $\alpha: T \rightarrow S$ and $\pi: S \rightarrow C(\mathcal{H})$ be continuous morphisms of semitopological semigroups, and*

$$(\forall x, y \in \mathcal{H}) \quad \psi_{x,y}^\pi: S \rightarrow \mathbb{C}, \quad \psi_{x,y}^\pi(\cdot) := (x \mid \pi(\cdot)y).$$

Then one has $\psi_{x,y}^\pi \in \mathcal{RUC}_b(S)$ and $\psi_{x,y}^\pi \circ \alpha = \psi_{x,y}^{\pi \circ \alpha} \in \mathcal{RUC}_b(T)$ for all $x, y \in \mathcal{H}$. If moreover π is continuous with respect to the strong operator topology on $C(\mathcal{H})$, then one also has $\psi_{x,y}^\pi \in \mathcal{LUC}_b(S)$ and $\psi_{x,y}^\pi \circ \alpha = \psi_{x,y}^{\pi \circ \alpha} \in \mathcal{LUC}_b(T)$ for all $x, y \in \mathcal{H}$.*

Proof. For all $s \in S$ we have $L_s(\psi_{x,y}^\pi) = \psi_{\pi(s)^*x, y}^\pi$ and $R_s(\psi_{x,y}^\pi) = \psi_{x, \pi(s)y}^\pi$ and then

$$\|L_s(\psi_{x,y}^\pi) - \psi_{x,y}^\pi\|_\infty = \|\psi_{\pi(s)^*x-x, y}^\pi\|_\infty \leq \|\pi(s)^*x - x\| \cdot \|y\|$$

and similarly

$$\|R_s(\psi_{x,y}^\pi) - \psi_{x,y}^\pi\|_\infty \leq \|x\| \cdot \|\pi(s)y - y\|.$$

Thus, using the continuity of the representation π , we obtain $\psi_{x,y}^\pi \in \mathcal{RUC}_b(S)$. Then the remaining assertions follow directly. We only recall that continuity of π with respect to the strong* operator topology on $C(\mathcal{H})$ means that

$$\lim_{s \rightarrow s_0} \|\pi(s)x - \pi(s_0)x\| = \lim_{s \rightarrow s_0} \|\pi(s)^*x - \pi(s_0)^*x\| = 0$$

for all $x \in \mathcal{H}$ and $s_0 \in S$. □

We use the following terminology. A *semigroup representation* is any semigroup morphism $\pi: S \rightarrow \mathcal{B}(\mathcal{H})$, where \mathcal{H} is a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ is regarded as multiplicative semigroup. We then define

$$\pi(S)' := \{a \in \mathcal{B}(\mathcal{H}) \mid (\forall s \in S) \pi(s)a = a\pi(s)\},$$

and we say that π is a *factor representation* if $\{0\}$ and \mathcal{H} are the only closed linear subspaces that are invariant to all operators from $\pi(S) \cup \pi(S)'$. It is clear that every unitary irreducible representation of a group is a factor representation. More generally, if for every $a \in \pi(G)$ we have $a^* \in \pi(G)$, then $\pi(G)''$ is a von Neumann algebra, and π is a factor representation if and only if $\pi(G)''$ is a factor in the sense of the theory of von Neumann algebras, that is, the center of $\pi(G)''$ is equal to the set of all scalar multiples of the identity operator on \mathcal{H} .

Lemma 3.2. *Let $\pi: S \rightarrow \mathcal{B}(\mathcal{H})$ be any semigroup representation, and for any normal subsemigroup $N_0 \subseteq S$ define $\mathcal{H}_0 := \{x \in \mathcal{H} \mid (\forall n \in N_0) \pi(n)x = x\}$. Then \mathcal{H}_0 is a linear subspace of \mathcal{H} that is invariant to any operator from $\pi(S) \cup \pi(S)'$.*

Proof. It is clear that if $a \in \mathcal{B}(\mathcal{H})$ and $\pi(s)a = a\pi(s)$ for every $s \in S$, then $a\mathcal{H}_0 \subseteq \mathcal{H}_0$. Moreover, for arbitrary $s \in S$ and $n \in N_0$ we have $ns = sn_1$ for some $n_1 \in N_0$, by the definition of the fact that N_0 is a normal subsemigroup of S . Then for all $x \in \mathcal{H}_0$ we obtain $\pi(n)\pi(s)x = \pi(ns)x = \pi(sn_1)x = \pi(s)\pi(n_1)x = \pi(s)x$, and thus $\pi(s)\mathcal{H}_0 \subseteq \mathcal{H}_0$. We have thus proved that \mathcal{H}_0 is invariant both to $\pi(S)$ and to $\pi(S)'$. \square

With Lemmas 3.1–3.2 at hand, we now prove the following generalization of [Ko82, Satz 1].

Proposition 3.3. *Let S be a semitopological monoid, and \mathcal{H} be a complex Hilbert space with its contraction semigroup $C(\mathcal{H})$ regarded as a topological semigroup with the strong* operator topology. Assume that $\pi: S \rightarrow C(\mathcal{H})$ is a continuous morphism of monoids, which is also a factor representation.*

Then there exists a neighbourhood V of $\mathbf{1} \in S$ such that for every amenable semitopological semigroup N and every continuous normal morphism of semigroups $\iota: N \rightarrow S$ with $\iota(N) \subseteq V$ we have $N \subseteq \text{Ker}(\pi \circ \iota)$.

Proof. Let $x_0 \in \mathcal{H}$ be any vector with $\|x_0\| = 1$. Since $\pi: S \rightarrow C(\mathcal{H})$ is a morphism of monoids, one has $\pi(\mathbf{1}) = \mathbf{1}$. Then $(x_0 \mid \pi(\mathbf{1})x_0) = 1$, hence there exists a neighbourhood V of $\mathbf{1} \in S$, depending on x_0 and π and satisfying

$$(\forall s \in V) \quad \text{Re}(x_0 \mid \pi(s)x_0) \geq 1/2. \quad (3.1)$$

Now let N be any amenable semitopological semigroup and $\iota: N \rightarrow S$ be any continuous morphism for which $\iota(N)$ is a normal subsemigroup of S and $\iota(N) \subseteq V$. We will prove that $(\pi \circ \iota)(n) = \mathbf{1} \in \mathcal{B}(\mathcal{H})$ for every $n \in N$. To this end, we define the closed linear subspace of \mathcal{H} ,

$$\mathcal{H}_N := \{x \in \mathcal{H} \mid \pi(\iota(n))x = x\}$$

and we will prove that $\mathcal{H}_N = \mathcal{H}$. It follows by Lemma 3.2 applied for $N_0 := \iota(N)$ that \mathcal{H}_N is invariant both to $\pi(S) \cup \pi(S)'$. As π is a factor representation, the equality $\mathcal{H}_N = \mathcal{H}$ will follow as soon as we will have proved that $\mathcal{H}_N \neq \{0\}$.

For arbitrary $x \in \mathcal{H}$, the function $\varphi_x: N \rightarrow \mathbb{C}$, $\varphi_x(\cdot) := (x \mid \pi(\iota(\cdot))x_0)$, satisfies $\varphi_x \in \mathcal{UC}_b(N)$ by Lemma 3.1. Since the semigroup N is amenable, there exists a linear functional $\mu: \mathcal{UC}_b(N) \rightarrow \mathbb{C}$ with $\mu(\mathbf{1}) = 1$ and $\mu(L_n(\varphi_x)) = \mu(\varphi_x)$ for every $x \in \mathcal{H}$ and $n \in N$. One has $|\mu(\varphi_x)| \leq \sup_N |\varphi_x(\cdot)| \leq \|x\|$ and $x \mapsto \mu(\varphi_x)$ is a linear functional on \mathcal{H} , hence by Riesz' theorem there exists a unique vector $x_1 \in \mathcal{H}$ with $\mu(\varphi_x) = (x \mid x_1)$ for every $x \in \mathcal{H}$. We then obtain

$$(x \mid \pi(\iota(n))x_1) = (\pi(\iota(n))^*x \mid x_1) = \mu(\varphi_{\pi(\iota(n))^*x}).$$

It is easily checked that $\varphi_{\pi(\iota(n))^*x} = L_n(\varphi_x)$ hence, using the property $\mu(L_n(\varphi_x)) = \mu(\varphi_x)$ for every $n \in N$, we obtain by the above equalities

$$(x \mid \pi(\iota(n))x_1) = \mu(\varphi_x) = (x \mid x_1)$$

for all $x \in \mathcal{H}$, hence $\pi(\iota(n))x_1 = x_1$ for every $n \in N$. This shows that $x_1 \in \mathcal{H}_N$. Note that, by $\iota(N) \subseteq V$ and (3.1), we obtain $\text{Re} \varphi_{x_0} \geq 1/2$ on N , and then, since $\mu \geq 0$ and $\mu(\mathbf{1}) = 1$, we have $\text{Re}(x_0 \mid x_1) = \text{Re}(\mu(\varphi_{x_0})) \geq 1/2$. This shows that $x_1 \neq 0$, hence $\mathcal{H}_N \neq \{0\}$, and this completes the proof, as we explained above. \square

Theorem 3.4. *Let S be a semitopological monoid with a filter basis \mathcal{N} of normal submonoids converging to the identity. Assume that every element of \mathcal{N} is an amenable semitopological semigroup with its induced topology from S .*

Then for every factor representation $\pi: G \rightarrow C(\mathcal{H})$ which is continuous with respect to the strong operator topology there exist $N_0 \in \mathcal{N}$ satisfying the following equivalent conditions:*

- (i) *There exists a factor representation $\pi_0: G/N_0 \rightarrow C(\mathcal{H})$ which is continuous with respect to the strong* operator topology and satisfies $\pi = \pi_0 \circ p_{N_0}$.*
- (ii) *One has $N \subseteq \text{Ker } \pi$.*

Proof. It is clear that conditions (i) and (ii) from the statement are equivalent. Now let V be the neighborhood of $\mathbf{1} \in G$ given by Lemma 3.3. Since \mathcal{N} is converging to the identity, there exists $N_0 \in \mathcal{N}$ with $N_0 \subseteq V$. Using Lemma 3.3 for the inclusion map $\iota: N_0 \hookrightarrow G$ and the fact that N_0 is amenable, it follows that $N_0 \subseteq \text{Ker } \pi$, hence there exists a unique unitary representation $\pi_0: S/N_0 \rightarrow \mathcal{B}(\mathcal{H})$ with $\pi = \pi_0 \circ p_N$. Since $p_N: S \rightarrow S/N$ is a quotient map and π is continuous, it then follows that π_0 is continuous and is a factor representation, and this completes the proof. \square

Corollary 3.5. *Let G be a topological group with a filter basis \mathcal{N} of closed normal subgroups converging to the identity. Assume that every element of \mathcal{N} is an amenable topological group with its induced topology from G .*

Then for every factor representation $\pi: G \rightarrow U(\mathcal{H})$ which is continuous with respect to the strong operator topology there exist $N_0 \in \mathcal{N}$ and a factor representation $\pi_0: G/N_0 \rightarrow \mathcal{B}(\mathcal{H})$ with $\pi = \pi_0 \circ p_{N_0}$.

Proof. Use Theorem 3.4 and the fact if a unitary representation $\pi: G \rightarrow U(\mathcal{H})$ is continuous with respect to the strong operator topology, then it is actually continuous with respect to the strong* operator topology. \square

Remark 3.6. Corollary 3.5 is a generalization of several results from the earlier literature on unitary representations; see for instance [Lip72, Th. 2.1], [Mo72, Prop. 2.2], [Mi75], [Mag81, Cor. to Lemma 1], [Ko82, Folg. 3–4], where the elements of the filter basis of normal submonoids \mathcal{N} are amenable because they are either compact or solvable subgroups.

Corollary 3.7. *Let G be any locally compact group with a filter basis \mathcal{N} of compact normal subgroups converging to the identity. For every irreducible representation $\pi: G \rightarrow U(\mathcal{H})$ there exist $N_0 \in \mathcal{N}$ and an irreducible representation $\pi_0: G/N_0 \rightarrow U(\mathcal{H})$ with $\pi = \pi_0 \circ p_{N_0}$.*

Proof. This is a special case of Corollary 3.5 \square

Application to infinite direct products of amenable groups. We now construct the main class of non-locally-compact groups to which the above results can be applied. To this end we need the following proposition on amenability of any infinite direct product of amenable topological groups, for later use in the proof of Corollary 4.9. This is a result that we were not able to locate in the existing literature, although it has been long known that in the category of discrete groups an infinite direct product of amenable groups may not be amenable (see [Da57, page 517]). The point of Proposition 3.8 below is that the explanation of that seemingly pathological behavior is that an infinite direct product of discrete topological spaces is not a discrete topological space. Amenability does behave well with respect to

infinite direct products as soon as we endow any infinite product of discrete groups with its proper topology, which is an infinite direct product of discrete topologies.

Proposition 3.8. *If $\{G_j\}_{j \in J}$ is a family of amenable topological groups, then their direct product $G := \prod_{j \in J} G_j$ is also an amenable topological group.*

For proving Proposition 3.8 we need Lemmas 3.9 and 3.11 below.

Lemma 3.9. *Let $\{H_j\}_{j \in J}$ be any family of topological groups. For every finite subset $F \subseteq J$ we denote*

$$H_F := \prod_{j \in F} G_j \text{ and } p_F: H \rightarrow H_F, (h_j)_{j \in J} \mapsto (h_j)_{j \in F}.$$

We define

$$\mathcal{A} := \bigcup_F \mathcal{A}_F$$

where $\mathcal{A}_F := \{\psi \circ p_F \mid \psi \in \mathcal{LUC}_b(H_F)\}$ as F runs over all finite subsets of J .

Then \mathcal{A} is a dense subset of $\mathcal{LUC}_b(H)$.

Proof. For every finite subset $F \subseteq J$ we also denote

$$H^{(F)} := \prod_{j \in J \setminus F} H_j.$$

There exists a canonical isomorphism $H_F \simeq H/H^{(F)}$, and thus p_F can be identified with the quotient map $p_{H^{(F)}}: H \rightarrow H/H^{(F)}$. Moreover, we will need the canonical injective morphism

$$\iota_F: H_F \hookrightarrow H$$

defined by $\iota_F((h_j)_{j \in F}) = (g_j)_{j \in J}$, where $g_j := h_j$ if $j \in F$ and $g_j := \mathbf{1} \in H_j$ if $j \in J \setminus F$.

Now let $\varphi \in \mathcal{LUC}_b(H)$ and $\varepsilon > 0$ arbitrary. Then there exists a neighborhood V of $\mathbf{1} \in H$ with $|\varphi(g) - \varphi(h)| < \varepsilon$ for all $g, h \in H$ with $gh^{-1} \in V$. By the definition of the topology of H , there exist a finite subset $F \subseteq J$ and for every $j \in F$ there exists a neighborhood V_j of $\mathbf{1} \in H_j$ with

$$\left(\prod_{j \in F} V_j \right) \times H^{(F)} \subseteq V. \quad (3.2)$$

We will prove that

$$\|\varphi - (\varphi \circ \iota_F) \circ p_F\|_\infty \leq \varepsilon \quad (3.3)$$

and this will complete the proof, because $\varphi \in \mathcal{LUC}_b(H)$ implies $\varphi \circ \iota_F \in \mathcal{LUC}_b(H_F)$ (since ι_F is a homomorphism of topological groups) hence $(\varphi \circ \iota_F) \circ p_F \in \mathcal{A}$. In order to prove (3.3), we note that for arbitrary $g = (g_j)_{j \in J} \in H$ we have $(\iota_F \circ p_F)(g) = h := (h_j)_{j \in J}$, where $h_j = g_j$ if $j \in F$ and $h_j = \mathbf{1} \in G_j$ if $j \in J \setminus F$. Hence $gh^{-1} = (k_j)_{j \in J}$, where $k_j = \mathbf{1} \in V_j$ for $j \in F$ and $k_j \in G_j$ for $j \in J \setminus F$. Thus, using (3.2), we obtain $g((\iota_F \circ p_F)(g))^{-1} \in V$ for every $g \in H$. According to the way V was selected, we then obtain (3.3), and we are done. \square

Remark 3.10. It is easily seen that the method of proof of Lemma 3.9 allows us to obtain a more precise conclusion, namely that for every $\varphi \in \mathcal{LUC}_b(G)$ one has

$$\lim_F \|\varphi - (\varphi \circ \iota_F \circ p_F)\|_\infty = 0$$

where the limit is taken with respect to the upwards directed set of all finite subsets of J , partially ordered with respect to inclusion.

Lemma 3.11. *Let H be a topological group with a filter basis \mathcal{N} of closed normal subgroups. For every $N \in \mathcal{N}$ we assume the quotient group H/N is amenable, we denote by $p_N: H \rightarrow H/N$ the corresponding quotient map, and we define $\mathcal{A}_N := \{\psi \circ p_N \mid \psi \in \mathcal{LUC}_b(H/N)\}$. Also define $\mathcal{A} := \bigcup_{N \in \mathcal{N}} \mathcal{A}_N$ with its closure in $\mathcal{LUC}_b(H)$ denoted by $\overline{\mathcal{A}}$. Then $\overline{\mathcal{A}}$ is invariant to the left-translation operator $L_s: \mathcal{LUC}_b(H) \rightarrow \mathcal{LUC}_b(H)$ for every $s \in H$, and there exists a left invariant mean on $\overline{\mathcal{A}}$.*

Proof. Using that $p_N: H \rightarrow H/N$ is a surjective morphism of groups, it follows that $\mathcal{A}_N \subseteq \mathcal{LUC}_b(H)$ is a closed linear subspace that contains the constant functions and is invariant to the operator $L_s: \mathcal{LUC}_b(H) \rightarrow \mathcal{LUC}_b(H)$ for arbitrary $s \in H$. In fact, for all $x \in H$ we have

$$(L_s(\psi \circ p_N))(x) = (\psi \circ p_N)(sx) = \psi(p_N(s)p_N(x)) = (L_{p_N(s)}\psi)(p_N(x))$$

hence $L_s(\psi \circ p_N) = (L_{p_N(s)}\psi) \circ p_N$, and this shows that \mathcal{A}_N is invariant under L_s .

We denote by \mathfrak{M}_N the set of all linear functionals $\mu: \mathcal{LUC}_b(H) \rightarrow \mathbb{C}$ satisfying $0 \leq \mu(\varphi)$ if $0 \leq \varphi \in \mathcal{A}_N$; $\|\mu\| = \mu(\mathbf{1}) = 1$, where $\mathbf{1}$ is the constant function equal to 1 on H ; and $\mu \circ L_s = \mu$ on \mathcal{A}_N for all $s \in H$. It is clear that \mathfrak{M}_N is a weak*-compact subset of the unit ball of $\mathcal{LUC}_b(H)^*$. To see that $\mathfrak{M}_N \neq \emptyset$, we may select any left invariant mean $\mu_0: \mathcal{LUC}_b(H/N) \rightarrow \mathbb{C}$, since G/N is amenable. Then we define $\tilde{\mu}_0: \mathcal{A}_N \rightarrow \mathbb{C}$, $\tilde{\mu}_0(\psi \circ p_N) := \mu_0(\psi)$, and we use the Hahn-Banach theorem to find a linear functional $\mu: \mathcal{LUC}_b(H) \rightarrow \mathbb{C}$ with $\mu|_{\mathcal{A}_N} = \tilde{\mu}_0$ and $\|\mu\| = \|\tilde{\mu}_0\| = 1$. Then it is easily checked that $\mu \in \mathfrak{M}_N$, hence $\mathfrak{M}_N \neq \emptyset$.

We now recall that if $N_1, N_2 \in \mathcal{N}$ with $N_1 \supseteq N_2$, then we have a natural surjective morphism of groups $p_{N_2, N_1}: G/N_2 \rightarrow G/N_1$ satisfying $p_{N_1} = p_{N_2, N_1} \circ p_{N_2}$, and this directly implies that $\mathcal{A}_{N_1} \subseteq \mathcal{A}_{N_2}$ and then $\mathfrak{M}_{N_1} \supseteq \mathfrak{M}_{N_2}$. Thus $\{\mathfrak{M}_N\}_{N \in \mathcal{N}}$ is a family of weak*-compact nonempty subsets of the unit ball of $\mathcal{LUC}_b(H)^*$ and the intersection of every finite subfamily is nonempty. Since the unit ball of $\mathcal{LUC}_b(H)^*$ is weak*-compact by the Banach-Alaoglu theorem, it then follows that there exists $\mu \in \bigcap_{N \in \mathcal{N}} \mathfrak{M}_N$. Then for every $s \in S$ and $N \in \mathcal{N}$ we have $\mu \circ L_s = \mu$ on \mathcal{A}_N hence,

we obtain $\mu \circ L_s = \mu$ on \mathcal{A} . Thus μ is a left invariant mean on $\overline{\mathcal{A}}$, and this completes the proof. \square

Proof of Proposition 3.8. We first introduce notation similar to that used in the proofs of Lemmas 3.11 and 3.9. Thus, for every finite subset $F \subseteq J$ we denote

$$G_F := \prod_{j \in F} G_j, \quad G^{(F)} := \prod_{j \in J \setminus F} G_j, \quad p_F: G \rightarrow G_F, (g_j)_{j \in J} \mapsto (g_j)_{j \in F}.$$

One has the canonical isomorphism $G_F \simeq G/G^{(F)}$, and thus p_F can be identified with the quotient map $p_{G^{(F)}}: G \rightarrow G/G^{(F)}$. Moreover, there is the canonical injective morphism

$$\iota_F: G_F \hookrightarrow G$$

defined by $\iota_F((h_j)_{j \in F}) = (g_j)_{j \in J}$, where $g_j := h_j$ if $j \in F$ and $g_j := \mathbf{1} \in G_j$ if $j \in J \setminus F$. Using these maps we define

$$\mathcal{A} := \bigcup_F \mathcal{A}_F$$

where $\mathcal{A}_F := \{\psi \circ p_F \mid \psi \in \mathcal{LUC}_b(G_F)\}$ as F runs over all finite subsets of J .

We now apply Lemma 3.11 for $H := G$ and

$$\mathcal{N} := \{G^{(F)} \mid F \subseteq J, |F| < \infty\}.$$

For every finite subset $F \subseteq J$ the group $G/G^{(F)} \simeq G_F$ is a direct product of finitely many amenable groups, hence it is well known that G_F is amenable. Thus the hypotheses of Lemma 3.11 are fulfilled, and we obtain a left invariant mean on $\overline{\mathcal{A}}$. On the other hand, one has $\overline{\mathcal{A}} = \mathcal{LUC}_b(G)$ by Lemma 3.9. It follows that there exists a left invariant mean on G , hence G is amenable, and we are done. \square

Remark 3.12. The proof of Proposition 3.8 uses some ideas from the proof of [Ko82, Lemma 1], which seems however to need some extra arguments, because it is not clear how to generalize the above Lemma 3.9 from infinite direct products to arbitrary projective limits. In this connection, it would be interesting to see if every projective limit of amenable topological groups is again an amenable topological group with respect to its projective limit topology. Just as in the special case of infinite direct products of discrete groups, it has long been known that a projective limit of amenable discrete groups may not be amenable as a discrete group (see [Da57, page 517] again).

Now we can draw the following consequence of Corollary 3.5.

Corollary 3.13. *Let $\{G_j\}_{j \in J}$ be any family of amenable topological groups, with their direct product $G := \prod_{j \in J} G_j$. For every factor representation $\pi: G \rightarrow U(\mathcal{H})$ which is continuous with respect to the strong operator topology there exists a finite subset $F \subseteq J$ and a factor representation $\pi_0: G_F \rightarrow \mathcal{B}(\mathcal{H})$ with $\pi = \pi_0 \circ p_F$, where $G_F := \prod_{j \in F} G_j$ and $p_F: G \rightarrow G_F$ is the natural projection.*

Proof. Use Corollary 3.5 along with Proposition 3.8. \square

4. COADJOINT ORBITS AND REPRESENTATIONS

In this section we study coadjoint orbits of pro-Lie groups and their corresponding unitary representations. The first part of the following definition is suggested by a similar, more general, notion of topological group with Lie algebra (see [HM07, Def. 2.11]). We choose to work here only with topological groups whose Lie algebras are locally convex because this class is general enough for our purposes and, on the other hand, the local convexity assumption ensures that the dual spaces of these Lie algebras are nontrivial, hence it is meaningful to study their corresponding coadjoint actions.

Definition 4.1. Let G be any topological group and endow its set of continuous 1-parameter subgroups,

$$\mathbf{L}(G) := \{X \in \mathcal{C}(\mathbb{R}, G) \mid (\forall t, s \in \mathbb{R}) \quad X(t+s) = X(t)X(s)\}.$$

with the topology of uniform convergence on the compact subsets of \mathbb{R} .

We say that G is a *topological group with locally convex Lie algebra* if the topological space $\mathbf{L}(G)$ has the structure of a locally convex Lie algebra over \mathbb{R} , whose scalar multiplication, vector addition and bracket satisfy the following conditions

for all $t, s \in \mathbb{R}$ and $X_1, X_2 \in \mathbf{L}(G)$:

$$\begin{aligned} (t \cdot X_1)(s) &= X_1(ts); \\ (X_1 + X_2)(t) &= \lim_{n \rightarrow \infty} (X_1(t/n)X_2(t/n))^n; \\ [X_1, X_2](t^2) &= \lim_{n \rightarrow \infty} (X_1(t/n)X_2(t/n)X_1(-t/n)X_2(-t/n))^{n^2}, \end{aligned}$$

where the convergence is assumed to be uniform on the compact subsets of \mathbb{R} . If this is the case, then we define

$$\mathbf{L}(G)^* := \{\xi: \mathbf{L}(G) \rightarrow \mathbb{R} \mid \xi \text{ is linear and continuous}\}$$

and we regard $\mathbf{L}(G)^*$ as a locally convex real vector space, endowed with its weak dual topology, that is, the topology of pointwise convergence on $\mathbf{L}(G)$.

We recall (see for instance [HM07] or [BNi15]) the *adjoint action*

$$\text{Ad}_G: G \times \mathbf{L}(G) \rightarrow \mathbf{L}(G), \quad (g, X) \mapsto \text{Ad}_G(g)X := gX(\cdot)g^{-1}$$

and this defines by duality the *coadjoint action*

$$\text{Ad}_G^*: G \times \mathbf{L}(G)^* \rightarrow \mathbf{L}(G)^*, \quad (g, \xi) \mapsto \text{Ad}_G^*(g)\xi := \xi \circ \text{Ad}_G(g^{-1}).$$

We denote by $\mathbf{L}(G)^*/G$ the set of all coadjoint orbits, that is, the orbits of the above coadjoint action.

Example 4.2. Every connected locally compact group is a topological group with locally convex Lie algebra (see for instance [Gl57] or [La57]).

Lemma 4.3. *In Definition 4.1, the coadjoint action is well defined.*

Proof. We must prove that if $\xi \in \mathbf{L}(G)^*$ and $g \in G$ then $\text{Ad}_G^*(g)\xi \in \mathbf{L}(G)^*$, that is, the function $\text{Ad}_G^*(g)\xi: \mathbf{L}(G) \rightarrow \mathbb{R}$ is indeed continuous and linear. Since $\text{Ad}_G(g)$ is continuous (see [HM07, Prop. 2.28]), the function $\text{Ad}_G^*(g)\xi$ is continuous on $\mathbf{L}(G)$. To prove that $\text{Ad}_G(g)$ is linear, we first note that for every $t \in \mathbb{R}$ the evaluation map $\mathbf{L}(G) \rightarrow G, X \mapsto X(t)$, is continuous. Then for all $X, Y \in \mathbf{L}(G)$ and $t \in \mathbb{R}$ we may write

$$\begin{aligned} (\text{Ad}_G(g)(X + Y))(t) &= \text{Ad}_G(g)((X + Y)(t)) \\ &= \text{Ad}_G(g)\left(\lim_{n \rightarrow \infty} (X(t/n)Y(t/n))^n\right) \\ &= \lim_{n \rightarrow \infty} (\text{Ad}_G(g)X(t/n)\text{Ad}_G(g)Y(t/n))^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n}\text{Ad}_G(g)X(t), \frac{1}{n}\text{Ad}_G(g)Y(t)\right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n}(\text{Ad}_G(g)X), \frac{1}{n}\text{Ad}_G(g)Y\right)^n(t) \\ &= (\text{Ad}_G(g)X + \text{Ad}_G(g)Y)(t) \\ &= \text{Ad}_G(g)X(t) + \text{Ad}_G(g)Y(t) \end{aligned}$$

Moreover, for all $s, t \in \mathbb{R}$ and $X \in \mathbf{L}(G)$, one has

$$\text{Ad}_G(g)(sX)(t) = \text{Ad}_G(g)(X)(st) = (s\text{Ad}_G(g)(X))(t).$$

Thus $\text{Ad}_G(g)$ is linear and for all $\xi \in \mathbf{L}(G)^*$, the function $\text{Ad}_G^*(g)\xi = \xi \circ \text{Ad}_G(g^{-1})$ is linear. This completes the proof. \square

The result of the above Lemma 4.3 should be regarded in the context of linearity properties of differentials on topological groups (see for instance [BB11] and the references therein).

Lemma 4.4. *Let $p: G_1 \rightarrow G_2$ be a continuous surjective morphism of topological groups with Lie algebras. If $\mathcal{O}_2 \in \mathbf{L}(G_2)^*/G_2$ is a coadjoint orbit of G_2 , then the set $\mathcal{O}_1 := \mathbf{L}(p)^*(\mathcal{O}_2) \subseteq \mathbf{L}(G_1)^*$ is a coadjoint orbit of G_1 .*

Proof. One has

$$\mathcal{O}_1 = \{\xi \circ \mathbf{L}(p) \mid \xi \in \mathcal{O}_2\}. \quad (4.1)$$

Let us fix $\xi_2 \in \mathcal{O}_2$ and denote $\xi_1 := \xi_2 \circ \mathbf{L}(p) \in \mathcal{O}_1$. We must prove that the coadjoint action of G_1 on \mathcal{O}_1 is transitive, hence that for arbitrary $\eta_1 \in \mathcal{O}_1$ there exists $g_1 \in G_1$ with $\xi_1 \circ \text{Ad}_{G_1}(g_1) = \eta_1$.

Since $\eta_1 \in \mathcal{O}_1$, it follows by (4.1) that there exists $\eta_2 \in \mathcal{O}_2$ with $\eta_2 \circ \mathbf{L}(p) = \eta_1$. But \mathcal{O}_2 is a coadjoint orbit of G_2 and $\xi_2, \eta_2 \in \mathcal{O}_2$, hence there exists $g_2 \in G_2$ with $\eta_2 = \xi_2 \circ \text{Ad}_{G_2}(g_2)$.

As $p: G_1 \rightarrow G_2$ is surjective, there exists $g_1 \in G_1$ with $p(g_1) = g_2$, and then $\eta_2 = \xi_2 \circ \text{Ad}_{G_2}(p(g_1))$, which further implies

$$\eta_2 \circ \mathbf{L}(p) = \xi_2 \circ \text{Ad}_{G_2}(p(g_1)) \circ \mathbf{L}(p): \mathbf{L}(G_1) \rightarrow \mathbb{R}. \quad (4.2)$$

On the other hand, every $X \in \mathbf{L}(G_1)$ is a function $X: \mathbb{R} \rightarrow G_1$ and we have $(\mathbf{L}(p))(X) \in \mathbf{L}(G_2)$ defined by $(\mathbf{L}(p))(X) := p \circ X$, hence

$$\begin{aligned} (\text{Ad}_{G_2}(p(g_1)) \circ \mathbf{L}(p))(X) &= p(g_1)((p \circ X)(\cdot))p(g_1)^{-1} \\ &= p \circ (g_1 X(\cdot)g_1^{-1}) \\ &= \mathbf{L}(p)((\text{Ad}_{G_1}(g_1))(X)). \end{aligned}$$

We thus obtain the commutative diagram

$$\begin{array}{ccc} \mathbf{L}(G_1) & \xrightarrow{\text{Ad}_{G_1}(g_1)} & \mathbf{L}(G_1) \\ \mathbf{L}(p) \downarrow & & \downarrow \mathbf{L}(p) \\ \mathbf{L}(G_2) & \xrightarrow{\text{Ad}_{G_2}(p(g_1))} & \mathbf{L}(G_2) \end{array}$$

for $g_1 \in G_1$. Then (4.2) is equivalent to $\eta_2 \circ \mathbf{L}(p) = \xi_2 \circ \mathbf{L}(p) \circ \text{Ad}_{G_1}(g_1)$, hence $\eta_1 = \xi_1 \circ \text{Ad}_{G_1}(g_1)$, and this completes the proof. \square

Lemma 4.5. *Let G be a topological group with a filter basis \mathcal{N} of closed normal subgroups converging to the identity. Assume that for every $N \in \mathcal{N}$ the quotient G/N is a Lie group.*

Then the following assertions hold:

- (i) *For every $N \in \mathcal{N}$ the linear map $\mathbf{L}(p_N)^*: \mathbf{L}(G/N)^* \rightarrow \mathbf{L}(G)^*$ is injective and moreover*

$$\mathbf{L}(G)^* = \bigcup_{N \in \mathcal{N}} \mathbf{L}(p_N)^*(\mathbf{L}(G/N)^*).$$

- (ii) *A set $\mathcal{O} \subseteq \mathbf{L}(G)^*$ is a coadjoint G -orbit if and only if there exist $N \in \mathcal{N}$ and a coadjoint (G/N) -orbit $\mathcal{O}_0 \subseteq \mathbf{L}(G/N)^*$ with $\mathbf{L}(p_N)^*(\mathcal{O}_0) = \mathcal{O}$.*

Proof. For every $N \in \mathcal{N}$ the map $\mathbf{L}(p_N): \mathbf{L}(G) \rightarrow \mathbf{L}(G/N)$ is surjective by [HM07, Lemma 4.19], hence the dual map $\mathbf{L}(p_N)^*: \mathbf{L}(G/N)^* \rightarrow \mathbf{L}(G)^*$ is injective. Similarly, if $N_1, N_2 \in \mathcal{N}$ with $N_1 \subseteq N_2$, then the map $\mathbf{L}(p_{N_1, N_2}): \mathbf{L}(G/N_1) \rightarrow \mathbf{L}(G/N_2)$

is surjective, and we thus obtain the projective limit of finite-dimensional Lie algebras $\{\mathbf{L}(G/N)\}_{N \in \mathcal{N}}$ whose projective limit is the locally convex Lie algebra $\mathbf{L}(G)$. It follows by [Kö69, §22, no. 6, Th. (6)] that $\mathbf{L}(G)^*$ is the inductive limit of the inductive system of finite-dimensional vector spaces $\{\mathbf{L}(G/N)^*\}_{N \in \mathcal{N}}$ connected by the linear maps $\mathbf{L}(p_{N_1, N_2})^*: \mathbf{L}(G/N_2)^* \rightarrow \mathbf{L}(G/N_1)^*$ for all $N_1, N_2 \in \mathcal{N}$ with $N_1 \subseteq N_2$, with the canonical maps $\mathbf{L}(p_N)^*: \mathbf{L}(G/N)^* \rightarrow \mathbf{L}(G)^*$ for all $N \in \mathcal{N}$. This completes the proof of the first assertion.

For the second assertion, it follows by Lemma 4.4 that for every $N \in \mathcal{N}$ and a coadjoint (G/N) -orbit $\mathcal{O}_0 \subseteq \mathbf{L}(G/N)^*$, the set $\mathbf{L}(p_N)^*(\mathcal{O}_0) \subseteq \mathbf{L}(G)^*$ is a coadjoint G -orbit. Conversely, let $\mathcal{O} \subseteq \mathbf{L}(G)^*$ be any coadjoint G -orbit. Select any $\xi \in \mathcal{O}$. Then it follows by the preceding assertion that there exists $N \in \mathcal{N}$ with $\xi \in \mathbf{L}(p_N)^*(\mathbf{L}(G/N)^*)$, that is, there exists $\xi_0 \in \mathbf{L}(G/N)^*$ with $\xi = \mathbf{L}(p_N)^*(\xi_0) = \xi_0 \circ \mathbf{L}(p_N)$. Let $\mathcal{O}_0 \subseteq \mathbf{L}(G/N)^*$ be the coadjoint orbit of ξ_0 . It follows by Lemma 4.4 again that the set $\mathbf{L}(p_N)^*(\mathcal{O}_0) \subseteq \mathbf{L}(G)^*$ is a coadjoint G -orbit. But we have $\xi = \mathbf{L}(p_N)^*(\xi_0) \in \mathbf{L}(p_N)^*(\mathcal{O}_0) \cap \mathcal{O}$, hence $\mathcal{O} = \mathbf{L}(p_N)^*(\mathcal{O}_0)$, because two coadjoint orbits are either disjoint from each other, or equal. This completes the proof. \square

We are now in a position to obtain one of the main results of the present paper, which extends Kirillov's correspondence beyond the category of Lie groups. Here we use the notation $\mathbf{L}(\cdot)_{\mathbb{Z}}^*$ for the set of all integral linear functionals on the dual of the Lie algebra of any connected nilpotent Lie group. (See Lemma A.2 below.)

Theorem 4.6. *Let G be a complete topological group with a filter basis \mathcal{N} of closed normal subgroups converging to the identity. Assume that for every $N \in \mathcal{N}$ the quotient G/N is a connected nilpotent Lie group and the topological group N is amenable.*

Then there exists a well-defined bijective correspondence

$$\Psi_G: \widehat{G} \rightarrow \mathbf{L}(G)^*/G, \quad [\pi] \mapsto \mathcal{O}^\pi$$

between the equivalence classes of unitary irreducible representations of G and the set of all coadjoint G -orbits contained in the G -invariant set

$$\mathbf{L}(G)_{\mathbb{Z}}^* := \{\xi \in \mathbf{L}(G)^* \mid (\exists N \in \mathcal{N})(\exists \eta \in \mathbf{L}(G/N)_{\mathbb{Z}}^*) \quad \xi = \eta \circ \mathbf{L}(p_N)\}.$$

Via the aforementioned correspondence, every unitary irreducible representation $\pi: G \rightarrow \mathcal{B}(\mathcal{H})$ is associated to the coadjoint G -orbit $\mathcal{O}^\pi := \mathbf{L}(p_N)^(\mathcal{O}_0) \subseteq \mathbf{L}(G)_{\mathbb{Z}}^*$, where $N \in \mathcal{N}$ and $\mathcal{O}_0 \subseteq \mathbf{L}(G/N)_{\mathbb{Z}}^*$ is the coadjoint (G/N) -orbit associated with a unitary irreducible representation $\pi_0: G/N \rightarrow \mathcal{B}(\mathcal{H})$ satisfying $\pi_0 \circ p_N = \pi$.*

Proof. We recall from Remark 2.2 that $G = \varprojlim_{N \in \mathcal{N}} G/N$, which implies $\mathbf{L}(G) =$

$\varprojlim_{N \in \mathcal{N}} \mathbf{L}(G/N)$. Then

$$\mathbf{L}(G)_{\mathbb{Z}}^* = \varinjlim_{N \in \mathcal{N}} \mathbf{L}(G/N)_{\mathbb{Z}}^*$$

is realized as an inductive limit of the corresponding sets $\mathbf{L}(G/N)_{\mathbb{Z}}^*$ where G/N is a connected nilpotent Lie group, and in particular

$$\mathbf{L}(G)_{\mathbb{Z}}^* = \bigcup_{N \in \mathcal{N}} \mathbf{L}(p_N)^* \mathbf{L}(G/N)_{\mathbb{Z}}^*$$

where $\mathbf{L}(p_N)^*: \mathbf{L}(G/N)_{\mathbb{Z}}^* \rightarrow \mathbf{L}(G)_{\mathbb{Z}}^*$ is the dual map of the canonical projection $p_N: G \rightarrow G/N$. This amounts to the following: For any coadjoint G -orbit $\mathcal{O} \subseteq$

$\mathbf{L}(G)_{\mathbb{Z}}^*$, there exist $N_0 \in \mathcal{N}$ and a coadjoint G/N_0 -orbit $\mathcal{O}_0 \subseteq \mathbf{L}(G/N_0)_{\mathbb{Z}}^*$ such that $\mathcal{O} = \mathbf{L}(p_{N_0})^*(\mathcal{O}_0)$.

Given π as in the statement, we use Corollary 3.5 in order to find $N \in \mathcal{N}$ and $\pi_0: G/N \rightarrow \mathcal{B}(\mathcal{H})$ with $\pi_0 \circ p_N = \pi$. Then, using Lemma A.2 for the connected nilpotent Lie group G/N , we find the coadjoint orbit $\mathcal{O}_0 \subseteq \mathbf{L}(G/N)_{\mathbb{Z}}^*$ associated with π_0 . Now $\mathcal{O}^\pi := \mathbf{L}(p_N)^*(\mathcal{O}_0) \subseteq \mathbf{L}(G)_{\mathbb{Z}}^*$ is a coadjoint G -orbit by Lemma 4.4.

We now prove that the coadjoint orbit \mathcal{O}^π does not depend on the choice of $N \in \mathcal{N}$ and $\pi_0: G/N \rightarrow \mathcal{B}(\mathcal{H})$ with $\pi_0 \circ p_N = \pi$, or, equivalently, $N \subseteq \text{Ker } \pi$ (see Theorem 3.4). To this end, let $N_1, N_2 \in \mathcal{N}$, $N_1 \cup N_2 \subseteq \text{Ker } \pi$, and $\pi_j: G/N_j \rightarrow \mathcal{B}(\mathcal{H})$ with $\pi_j \circ p_{N_j} = \pi$, and let $\mathcal{O}_j \subseteq \mathbf{L}(G/N_j)_{\mathbb{Z}}^*$ be the coadjoint orbit corresponding to π_j for $j = 1, 2$. We claim that $\mathbf{L}(p_{N_1})^*(\mathcal{O}_1) = \mathbf{L}(p_{N_2})^*(\mathcal{O}_2)$.

To prove that claim, first suppose that $N_1 \subseteq N_2$. Then one has the map $p_{N_1, N_2}: G/N_1 \rightarrow G/N_2$, $gN_1 \mapsto gN_2$, which is a continuous surjective homomorphism of connected nilpotent Lie groups, hence $\mathbf{L}(p_{N_1, N_2})^*: \mathbf{L}(G/N_2)_{\mathbb{Z}}^* \rightarrow \mathbf{L}(G/N_1)_{\mathbb{Z}}^*$ is injective. Since $p_{N_1, N_2} \circ p_{N_1} = p_{N_2}$, it follows that $\pi_2 \circ p_{N_1, N_2} = \pi_1$ by the construction from the proof of Theorem 3.4. Then, by Proposition A.3, we obtain $\mathbf{L}(p_{N_1, N_2})^*(\mathcal{O}_2) = \mathcal{O}_1$. Using again the equality $p_{N_1, N_2} \circ p_{N_1} = p_{N_2}$, one has $\mathbf{L}(p_{N_1})^* \circ \mathbf{L}(p_{N_1, N_2})^* = \mathbf{L}(p_{N_2})^*$, hence

$$\mathbf{L}(p_{N_1})^*(\mathcal{O}_1) = \mathbf{L}(p_{N_1})^* \mathbf{L}(p_{N_1, N_2})^*(\mathcal{O}_2) = \mathbf{L}(p_{N_2})^*(\mathcal{O}_2)$$

as claimed above.

We thus proved that the correspondence Ψ_G from the statement is well defined. This correspondence is surjective. In fact, let \mathcal{O} be any coadjoint G -orbit. By Lemma 4.5 there exist $N \in \mathcal{N}$ and a coadjoint G/N -orbit \mathcal{O}_0 with $\mathbf{L}(p_N)^*(\mathcal{O}_0) = \mathcal{O}$. Then, using Lemma A.2 for the nilpotent Lie group G/N , we find the unitary representation π_0 associated to \mathcal{O}_0 . Now, $\pi = \pi_0 \circ p_N$ is a unitary representation of G associated with the coadjoint G -orbit \mathcal{O} .

For injectivity, let \mathcal{O} and \mathcal{O}' be two coadjoint G -orbits. By Lemma 4.5 (and its proof), there exist $N \in \mathcal{N}$ and two (G/N) -orbits \mathcal{O}_0 and \mathcal{O}'_0 with $\mathcal{O} = \mathbf{L}(p_N)^*(\mathcal{O}_0)$ and $\mathcal{O}' = \mathbf{L}(p_N)^*(\mathcal{O}'_0)$. Since the map $\mathbf{L}(p_N)^*$ is injective, one has $\mathcal{O} \neq \mathcal{O}'$ if and only if $\mathcal{O}_0 \neq \mathcal{O}'_0$, and by Lemma A.2, this is further equivalent to the fact that \mathcal{O}_0 and \mathcal{O}'_0 are associated respectively with inequivalent unitary representations π_0 and π'_0 , and that inequivalence is the same thing with inequivalence of the representations hence $\pi = \pi_0 \circ p_N$ and $\pi'_0 \circ p_N = \pi'$. This completes the proof. \square

A version of Theorem 4.6 in which the quotients G/N are simply connected is stated in [Ko82, Satz 2], however, the corresponding proof seems to be incomplete.

In the setting of Theorem 4.6, the bijective correspondence from the equivalence classes of unitary irreducible representations of G to the coadjoint orbits of G can be regarded as a generalized Kirillov correspondence for G . We also note that, as a direct consequence of Theorem 4.6, the G -invariant set $\mathbf{L}(G)_{\mathbb{Z}}^*$ of “integral functionals” is actually independent on the choice of the filter basis \mathcal{N} satisfying the conditions from the statement of the theorem.

Corollary 4.7. *If G is a connected locally compact nilpotent group, then there are a G -invariant subset $\mathbf{L}(G)_{\mathbb{Z}}^*$ and a bijective correspondence $\Psi_G: \widehat{G} \rightarrow \mathbf{L}(G)_{\mathbb{Z}}^*/G$ onto the set of all coadjoint G -orbits contained in $\mathbf{L}(G)_{\mathbb{Z}}^*$. Moreover, if \mathcal{N} is a filter basis of closed normal subgroups of G converging to the identity for which G/N is a Lie*

group for every $N \in \mathcal{N}$, then

$$\mathbf{L}(G)_{\mathbb{Z}}^* := \{\xi \in \mathbf{L}(G)^* \mid (\exists N \in \mathcal{N})(\exists \eta \in \mathbf{L}(G/N)_{\mathbb{Z}}^*) \quad \xi = \eta \circ \mathbf{L}(p_N)\}.$$

Proof. Using Remark 2.5, there exists a filter basis \mathcal{N} of closed normal subgroups of G converging to the identity for which G/N is a Lie group if $N \in \mathcal{N}$. Moreover, since G is a nilpotent group and is connected, it follows that for every $N \in \mathcal{N}$ the quotient G/N is a connected nilpotent Lie group. On the other hand, N is also nilpotent, hence amenable. Thus the assertion follows by Theorem 4.6. \square

Example 4.8. Let us revisit Example 2.4 from the present perspective. Assume that \mathfrak{g} is a finite-dimensional nilpotent Lie algebra with its corresponding Lie group $\tilde{G} = (\mathfrak{g}, \cdot)$ defined by the Baker-Campbell-Hausdorff multiplication. Then \tilde{G} is a connected, simply connected, nilpotent Lie group, whose exponential map is the identity map of \mathfrak{g} . Denote by \mathfrak{z} the center of \mathfrak{g} , and let $n := \dim \mathfrak{z}$, so that we may assume $\mathfrak{z} = \mathbb{R}^n$. Then $\tilde{Z} = (\mathfrak{z}, +)$ is the center of \tilde{G} , and let us define the lattice $\Gamma_1 := \mathbb{Z}^n \subset \mathbb{R}^n = \mathfrak{z}$. Now select any sequence of discrete subgroups $(\mathbb{Z}^n, +) = \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$ with $\bigcap_{k \geq 1} \Gamma_k = \{0\}$, and with the corresponding projective system of Lie groups $\tilde{G}/\Gamma_1 \rightarrow \tilde{G}/\Gamma_2 \rightarrow \dots$. Define

$$G := \varprojlim_{k \geq 1} \tilde{G}/\Gamma_k \hookrightarrow \prod_{k \geq 1} \tilde{G}/\Gamma_k.$$

It follows by [Cz74, Lemma and Th. 2] that G is a locally compact group and one has a canonical isomorphism of Lie algebras

$$\mathfrak{g} \simeq \mathbf{L}(G).$$

Moreover, G is nilpotent because it is isomorphic to a closed subgroup of the topological group $\prod_{k \geq 1} \tilde{G}/\Gamma_k$, which is easily seen to be nilpotent. Consequently, by

Corollary 4.7, there exists a bijection $\Psi_G: \hat{G} \rightarrow \mathbf{L}(G)^*/G$ onto the set of all coadjoint G -orbits contained in the set $\mathbf{L}(G)_{\mathbb{Z}}^*$ of “integral functionals” on $\mathbf{L}(G)$.

We now provide a precise description of $\mathbf{L}(G)_{\mathbb{Z}}^*$. For $k = 1, 2, \dots$ we have $\mathbf{L}(\tilde{G}/\Gamma_k) = \mathfrak{g}$, hence, by Lemma A.2, $\mathbf{L}(\tilde{G}/\Gamma_k)_{\mathbb{Z}}^* := \{\xi \in \mathfrak{g}^* \mid \xi(\Gamma_k) \subseteq \mathbb{Z}\}$. Thus, by Corollary 4.7,

$$\mathbf{L}(G)_{\mathbb{Z}}^* := \{\xi \in \mathfrak{g}^* \mid (\exists k \geq 1) \quad \xi(\Gamma_k) \subseteq \mathbb{Z}\}.$$

Corollary 4.9. Let $\{G_j\}_{j \in J}$ be any family of connected nilpotent Lie groups, with their direct product topological group $G := \prod_{j \in J} G_j$. Define

$$\mathbf{L}(G)_{\mathbb{Z}}^* := \{\xi \in \mathbf{L}(G)^* \mid (\exists F \in \mathcal{F})(\exists \eta \in \mathbf{L}(G_F)_{\mathbb{Z}}^*) \quad \xi = \eta \circ \mathbf{L}(p_F)\}$$

where \mathcal{F} is the set of all finite subsets $F \subseteq J$, and for every $F \in \mathcal{F}$ we define $G_F := \prod_{j \in F} G_j$ and $p_F: G \rightarrow G_F$ is the natural projection. Then $\mathbf{L}(G)_{\mathbb{Z}}^*$ is a G -invariant subset of $\mathbf{L}(G)$ and there is a bijective correspondence $\Psi_G: \hat{G} \rightarrow \mathbf{L}(G)^*/G$ onto the set of all coadjoint G -orbits contained in $\mathbf{L}(G)_{\mathbb{Z}}^*$.

Proof. For every $F \in \mathcal{F}$ the group $G^{(F)} := \prod_{j \in J \setminus F} G_j$ is amenable by Proposition 3.8 and can be naturally regarded as a closed normal subgroup of G . Moreover, one has the canonical isomorphism $G_F \simeq G/G^{(F)}$, and p_F can be identified with the

quotient map $p_{G(F)}: G \rightarrow G/G^{(F)}$. Since the group G_F is a connected nilpotent Lie group, we may apply Theorem 4.6, and we are done. \square

Example 4.10. Here we illustrate Corollary 4.9 by the simplest example, which already shows that the usual C^* -algebraic approach to group representation theory breaks down for infinite direct products of non-compact locally compact groups, hence the description of their unitary duals in terms of coadjoint orbits is particularly important whenever it is available, as it is the only description known so far.

Specifically, let $J := \mathbb{N} = \{0, 1, 2, \dots\}$ and $G_j = (\mathbb{R}, +)$ for every $j \in J$. Then $G = (\mathbb{R}^{\mathbb{N}}, +)$, $\mathbf{L}(G) = \mathbb{R}^{\mathbb{N}}$ is the vector space of all sequences of real numbers, and $\mathbf{L}(G)^* = \mathbb{R}^{(\mathbb{N})}$ is the vector space of all finitely supported sequences of real numbers, and the coadjoint orbits of G can be identified with the points of $\mathbf{L}(G)^*$ since G is an abelian group. It thus follows by Corollary 4.9 or by direct verification that there exists a bijection $\Psi_G: \widehat{G} \rightarrow \mathbb{R}^{(\mathbb{N})}$. However, as the vector space $\mathbb{R}^{(\mathbb{N})}$ is infinite dimensional, it is not locally compact, hence it is not homeomorphic to the spectrum of any C^* -algebra. Consequently, the irreducible representation theory of G can be described via coadjoint orbits but not via any C^* -algebra.

See however [Gr05], [GN13], and the references therein for an interesting C^* -algebraic approach to representation theory of topological groups that are not locally compact.

APPENDIX A. COMPLEMENTS ON REPRESENTATIONS OF NILPOTENT LIE GROUPS

Our basic references for representation theory of nilpotent Lie groups are [Ki62] and [CG90]. In this section we record a few results that are needed in the main body of our paper. For the reader's convenience we provide self-contained proofs for some of the results that we found more difficult to locate in the literature.

Lemma A.1. *Let $p: G_1 \rightarrow G_2$ be a continuous surjective morphism of connected and simply connected nilpotent Lie groups. Let $\pi_2: G_2 \rightarrow \mathcal{B}(\mathcal{H})$ be any unitary irreducible representation of G_2 , associated with a coadjoint orbit $\mathcal{O}_2 \subseteq \mathbf{L}(G_2)^*$. Then $\pi_1 := \pi_2 \circ p: G_1 \rightarrow \mathcal{B}(\mathcal{H})$ is a unitary irreducible representation of G_1 , and its corresponding coadjoint orbit is $\mathcal{O}_1 := \mathbf{L}(p)^*(\mathcal{O}_2) \subseteq \mathbf{L}(G_1)^*$.*

Proof. It follows by Lemma 4.4 that \mathcal{O}_1 is a coadjoint orbit of G_1 .

Let $\ell_2 \in \mathcal{O}_2$ and $\ell_1 = \mathbf{L}(p)^*(\ell_2)$. To construct π_2 , we select a real polarization \mathfrak{h}_2 in ℓ_2 , i.e. $\mathfrak{h}_2 \subset \mathbf{L}(G_2)$ such that:

1. \mathfrak{h}_2 is a subalgebra of $\mathbf{L}(G_2)$,
2. \mathfrak{h}_2 is subordinated to ℓ_2 , i.e. $\langle \ell_2, [\mathfrak{h}_2, \mathfrak{h}_2] \rangle = 0$,
3. \mathfrak{h}_2 is a maximal isotropic subspace, i.e.,

$$\dim \mathfrak{h}_2 = \frac{1}{2} \dim \mathbf{L}(G_2) + \frac{1}{2} \dim \mathbf{L}(G_2)(\ell_2).$$

We denote by $H_2 = \exp \mathfrak{h}_2$ the analytic subgroup with Lie algebra \mathfrak{h}_2 . The exponential of the character $i\ell_2$ of \mathfrak{h}_2 is a character χ_{ℓ_2} of H_2 , defined by:

$$\chi_{\ell_2}(\exp X) = e^{i\langle \ell_2, X \rangle} \quad (X \in \mathfrak{h}_2).$$

The representation π_2 is an induced unitary representation:

$$\pi_2 = \text{Ind}_{H_2}^{G_2} \chi_{\ell_2}.$$

It is realized in the completion of the space $\mathcal{C}^\infty(G_2 : H_2)$ of C^∞ functions φ on G_2 , with compact support modulo H_2 , satisfying

$$\varphi(xh) = \chi_{\ell_2}(h)^{-1} \varphi(x) \quad \text{for } x \in G_2, h \in H_2, \quad \text{and} \quad \|\varphi\|^2 = \int_{G_2/H_2} |\varphi(\dot{x})|^2 d\dot{x}$$

The representation π_2 is defined by:

$$(\pi_2(x)\varphi)(y) = \varphi(x^{-1}y), \quad \text{for } \varphi \in \mathcal{C}^\infty(G_2 : H_2).$$

First, the Lie algebra of $G_2 = p(G_1)$ is $\mathbf{L}(G_2) = \mathbf{L}(p)(\mathbf{L}(G_1))$, hence $\mathbf{L}(p)$ is surjective. Moreover

$$\begin{aligned} \mathbf{L}(G_1)(\ell_1) &:= \{X \in \mathbf{L}(G_1) \mid \text{ad}^*(X)\ell_1 = 0\} \\ &= \{X \in \mathbf{L}(G_1) \mid (\forall Y \in \mathbf{L}(G_1)) \langle (\mathbf{L}p)^*(\ell_2), [X, Y] \rangle = 0\} \\ &= \{X \in \mathbf{L}(G_1) \mid (\forall Y \in \mathbf{L}(G_1)) \langle \ell_2, [\mathbf{L}(p)(X), \mathbf{L}(p)(Y)] \rangle = 0\} \\ &= \{X \in \mathbf{L}(G_1) \mid \mathbf{L}(p)(X) \in \mathbf{L}(G_2)(\ell_2)\} \\ &= (\mathbf{L}(p))^{-1}(\mathbf{L}(G_2)(\ell_2)). \end{aligned}$$

This implies that $\text{Ker } \mathbf{L}(p) \subset \mathbf{L}(G_1)(\ell_1)$, $\mathbf{L}(p)|_{\mathbf{L}(G_1)(\ell_1)} : \mathbf{L}(G_1)(\ell_1) \rightarrow \mathbf{L}(G_2)(\ell_2)$ is surjective, and

$$\dim \mathbf{L}(G_2)(\ell_2) + \dim(\text{Ker } \mathbf{L}(p)) = \dim(\mathbf{L}(G_1)(\ell_1)).$$

Then, we put $\mathfrak{h}_1 := (\mathbf{L}(p))^{-1}(\mathfrak{h}_2)$. Since \mathfrak{h}_2 is a maximal isotropic subspace, one has $\mathbf{L}(G_2)(\ell_2) \subset \mathfrak{h}_2$, hence

$$\text{Ker } \mathbf{L}(p) \subset \mathbf{L}(G_1)(\ell_1) = (\mathbf{L}(p))^{-1}(\mathbf{L}(G_2)(\ell_2)) \subset (\mathbf{L}(p))^{-1}\mathfrak{h}_2 = \mathfrak{h}_1.$$

The same argument gives

$$\dim \mathfrak{h}_2 + \dim \text{Ker } \mathbf{L}(p) = \dim \mathfrak{h}_1, \quad \dim \mathbf{L}(G_2) + \dim \text{Ker } \mathbf{L}(p) = \dim \mathbf{L}(G_1).$$

Now \mathfrak{h}_1 is a polarization in ℓ_1 . In fact:

1. \mathfrak{h}_1 is a subalgebra of $\mathbf{L}(G_1)$: For all $X, Y \in \mathfrak{h}_1$,

$$\mathbf{L}(p)([X, Y]) = [\mathbf{L}(p)(X), \mathbf{L}(p)(Y)] \in [\mathfrak{h}_2, \mathfrak{h}_2] \subset \mathfrak{h}_2,$$

then $[X, Y] \in \mathfrak{h}_1$.

2. \mathfrak{h}_1 is subordinated to ℓ_1 :

$$\begin{aligned} \langle \ell_1, [\mathfrak{h}_1, \mathfrak{h}_1] \rangle &= \langle (\mathbf{L}p)^*\ell_2, [\mathfrak{h}_1, \mathfrak{h}_1] \rangle = \langle \ell_2, \mathbf{L}(p)([\mathfrak{h}_1, \mathfrak{h}_1]) \rangle \\ &= \langle \ell_2, [\mathbf{L}(p)(\mathfrak{h}_1), \mathbf{L}(p)(\mathfrak{h}_1)] \rangle \subset \langle \ell_2, [\mathfrak{h}_2, \mathfrak{h}_2] \rangle = 0. \end{aligned}$$

3. \mathfrak{h}_1 is a maximal isotropic subspace:

$$\begin{aligned} \dim \mathfrak{h}_1 &= \dim \mathfrak{h}_2 + \dim \text{Ker } \mathbf{L}(p) \\ &= \frac{1}{2}(\dim \mathbf{L}(G_2) + \dim \text{Ker } \mathbf{L}(p) + \dim \mathbf{L}(G_2)(\ell_2) + \dim \text{Ker } \mathbf{L}(p)) \\ &= \frac{1}{2}(\dim \mathbf{L}(G_1) + \dim \mathbf{L}(G_1)(\ell_1)). \end{aligned}$$

Then the representation associated to the coadjoint orbit \mathcal{O}_1 is (with the same notation as above):

$$\rho_1 = \text{Ind}_{H_1}^{G_1} \chi_{\ell_1}.$$

Note that as $\text{Ker } \mathbf{L}(p) \subset \mathfrak{h}_1$, one has $\text{Ker } p = \exp(\text{Ker } \mathbf{L}(p)) \subset \exp(\mathfrak{h}_1) = H_1$. Hence, $\forall \psi \in \mathcal{C}^\infty(G_1 : H_1)$, $x \in G_1$ and $k \in \text{Ker } p$, if $k = \exp X$ then

$$\begin{aligned} \psi(xk) &= \chi_{\ell_1}(k)^{-1} \psi(x) = e^{-i\langle \ell_1, X \rangle} \psi(x) = e^{-i\langle \mathbf{L}(p)^* \ell_2, X \rangle} \psi(x) \\ &= e^{-i\langle \ell_2, \mathbf{L}(p)(X) \rangle} \psi(x) = \psi(x). \end{aligned}$$

The function ψ passes to the quotient, i.e., there exists $\varphi: G_2 = G_1/\text{Ker } p \rightarrow \mathbb{C}$ such that $\varphi \circ p = \psi$. Moreover, if $h = \exp X \in H_2$, then $X \in \mathfrak{h}_2 = \mathbf{L}(p)(\mathfrak{h}_1)$, $X = \mathbf{L}(p)(Y)$, for each $x = p(y)$ in G_2 , we have:

$$\varphi(xh) = \varphi(p(y)p(\exp Y)) = \psi(y \exp Y) = e^{-i\langle \ell_1, Y \rangle} \psi(y) = e^{-i\langle \ell_2, X \rangle} \varphi(x).$$

In other words, $\varphi \in \mathcal{C}^\infty(G_2 : H_2)$. Conversely, if $\varphi \in \mathcal{C}^\infty(G_2 : H_2)$, then $\varphi \circ p \in \mathcal{C}^\infty(G_1 : H_1)$. Thus, there is a linear bijection $\mathcal{C}^\infty(G_2 : H_2) \rightarrow \mathcal{C}^\infty(G_1 : H_1)$, $\varphi \mapsto \varphi \circ p$.

On the other hand, we have:

$$G_2/H_2 \simeq (G_1/\text{Ker } p)/(H_1/\text{Ker } p) \simeq G_1/H_1,$$

and

$$\|\psi\|^2 = \int_{G_1/H_1} |\psi(\dot{x})|^2 d\dot{x} = \int_{G_2/H_2} |\varphi(\dot{x})|^2 d\dot{x} = \|\varphi\|^2.$$

The bijection above extends uniquely to a unitary operator $\mathcal{H}_{\pi_2} \rightarrow \mathcal{H}_{\rho_1}$.

Note that, if $x, y \in G_1$

$$\begin{aligned} (\rho_1(x)U\varphi)(y) &= (U\varphi)(x^{-1}y) = (\varphi \circ p)(x^{-1}y) = \varphi(p(x)^{-1}p(y)) \\ &= \left((\pi_2 \circ p)(x)(\varphi \circ p) \right)(y) = \left((\pi_2 \circ p)(x)(U\varphi) \right)(y), \end{aligned}$$

that is, $\rho_1(x) = (\pi_2 \circ p)(x) = \pi_1(x)$, $\rho_1 = \pi_1$, and this completes the proof. \square

Lemma A.2. *Let G be any connected and simply connected nilpotent Lie group with a discrete normal subgroup $\Gamma \subseteq G$, and denote by $p: G \rightarrow G/\Gamma$ the quotient map. Denote by $\log_G: \mathfrak{g} \rightarrow G$ the inverse of the exponential map of G . Then for every unitary irreducible representation $\pi: G/\Gamma \rightarrow \mathcal{B}(\mathcal{H})$ the representation $\pi \circ p: G \rightarrow \mathcal{B}(\mathcal{H})$ is irreducible, and its corresponding coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ has the property that for all $\xi \in \mathcal{O}$ one has $\xi(\log_G(\Gamma)) \subseteq \mathbb{Z}$. One thus obtains a bijective correspondence from $\widehat{G/\Gamma}$ onto the set of all coadjoint G -orbits contained in the G -invariant set $\mathfrak{g}_{\mathbb{Z}}^* := \{\xi \in \mathfrak{g}^* \mid \xi(\log_G(\Gamma)) \subseteq \mathbb{Z}\}$.*

Proof. See [Ki62, Th. 8.1] and [CG90, Th. 4.4.2]. \square

Now we can prove the following generalization of Lemma A.1 to connected nilpotent Lie groups that may not be simply connected.

Proposition A.3. *Let $p: G_1 \rightarrow G_2$ be a continuous surjective morphism of connected nilpotent Lie groups. Let $\pi_2: G_2 \rightarrow \mathcal{B}(\mathcal{H})$ be any unitary irreducible representation of G_2 , associated with a coadjoint orbit $\mathcal{O}_2 \subseteq \mathbf{L}(G_2)^*$. Then $\pi_1 := \pi_2 \circ p: G_1 \rightarrow \mathcal{B}(\mathcal{H})$ is a unitary irreducible representation of G_1 , and its corresponding coadjoint orbit is $\mathcal{O}_1 := \mathbf{L}(p)^*(\mathcal{O}_2) \subseteq \mathbf{L}(G_1)^*$.*

Proof. It follows by Lemma 4.4 that \mathcal{O}_1 is a coadjoint orbit of G_1 . Let \tilde{G}_j be the universal covering group of G_j , with a suitable discrete central subgroup Γ_j and the quotient map $p_j: \tilde{G}_j \rightarrow \tilde{G}_j/\Gamma_j = G_j$. Since \tilde{G}_1 is simply connected, it follows

that the Lie group morphism $p \circ p_1 : \tilde{G}_1 \rightarrow G_2$ lifts to a unique Lie group morphism $\tilde{p} : \tilde{G}_1 \rightarrow \tilde{G}_2$ for which the diagram

$$\begin{array}{ccc} \tilde{G}_1 & \xrightarrow{\tilde{p}} & \tilde{G}_2 \\ p_1 \downarrow & & \downarrow p_2 \\ G_1 & \xrightarrow{p} & G_2 \end{array} \quad (\text{A.1})$$

is commutative, hence $p_2 \circ \tilde{p} = p \circ p_1$. Since $\Gamma_j = \text{Ker } p_j$, it then follows that $\tilde{p}(\Gamma_1) = \Gamma_2$. Now the conclusion follows, using of Lemmas A.1 and A.2.

Specifically, since $p : G_1 \rightarrow G_2$ is surjective and π_2 is irreducible, it is easily checked that $\pi_2 \circ p$ is an irreducible representation of G_1 . In order to identify the coadjoint orbit of G_1 associated with $\pi_2 \circ p$ via Lemma A.2, we first note that

$$\pi_2 \circ p \circ p_1 = (\pi_2 \circ p_2) \circ \tilde{p}$$

by (A.1). The above equality implies by Lemma A.1 that if we denote by $\tilde{\mathcal{O}}_2 \subseteq \mathbf{L}(\tilde{G}_2)^* = \mathbf{L}(G_2)^*$ the coadjoint orbit of \tilde{G}_2 that is associated with $\pi_2 \circ p_2$, then

$$\tilde{\mathcal{O}}_1 := \mathbf{L}(\tilde{p})^*(\mathcal{O}_2) \subseteq \mathbf{L}(\tilde{G}_1)^* \quad (\text{A.2})$$

is the coadjoint orbit of \tilde{G}_1 that is associated with $\pi_2 \circ p \circ p_1$, where we have used the notation \mathcal{O}_2 introduced in the statement. On the other hand, for $j = 1, 2$, the group morphism $p_j : \tilde{G}_j \rightarrow G_j$ is a covering map, hence $\mathbf{L}(p_j) : \mathbf{L}(\tilde{G}_j) \rightarrow \mathbf{L}(G_j)$ is an isomorphism of Lie algebras. By Lemma A.2, the coadjoint orbit of G_2 associated with π_2 is just the coadjoint orbit of \tilde{G}_2 associated with $\pi_2 \circ p_2$. More precisely, using the vector space isomorphism $\mathbf{L}(p_2)^* : \mathbf{L}(G_2)^* \rightarrow \mathbf{L}(\tilde{G}_2)^*$, one has

$$\mathbf{L}(p_2)^*(\mathcal{O}_2) = \tilde{\mathcal{O}}_2. \quad (\text{A.3})$$

Similarly, by Lemma A.2 again, the coadjoint orbit of G_1 associated with $\pi_2 \circ p$ is just the coadjoint orbit of \tilde{G}_1 associated with $(\pi_2 \circ p) \circ p_1$. More precisely, using the vector space isomorphism $\mathbf{L}(p_1)^* : \mathbf{L}(G_1)^* \rightarrow \mathbf{L}(\tilde{G}_1)^*$, one has

$$\mathbf{L}(p_1)^*(\mathcal{O}_1) = \tilde{\mathcal{O}}_1. \quad (\text{A.4})$$

Also, by (A.1), one has $p \circ p_1 = p_2 \circ \tilde{p}$, hence $\mathbf{L}(p) \circ \mathbf{L}(p_1) = \mathbf{L}(p_2) \circ \mathbf{L}(\tilde{p})$, and then $\mathbf{L}(p_1)^* \circ \mathbf{L}(p)^* = \mathbf{L}(\tilde{p})^* \circ \mathbf{L}(p_2)^*$, which further implies by (A.3), (A.2), and (A.4),

$$\mathbf{L}(p_1)^*(\mathbf{L}(p)^*(\mathcal{O}_2)) = \mathbf{L}(\tilde{p})^*(\mathbf{L}(p_2)^*(\mathcal{O}_2)) = \mathbf{L}(\tilde{p})^*(\tilde{\mathcal{O}}_2) = \tilde{\mathcal{O}}_1 = \mathbf{L}(p_1)^*(\mathcal{O}_1).$$

Now, as $\mathbf{L}(p_1)^*$ is a vector space isomorphism, we obtain $\mathbf{L}(p)^*(\mathcal{O}_2) = \mathcal{O}_1$, as claimed, and this completes the proof. \square

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